TITLE: Shape-Measure Method for Introducing the Nearly Optimal Domain

Shape-measure method for introducing the nearly optimal domain

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Abstract

We deal with introducing a new algorithm for solving the optimal shape problems in which they are defined with respect to a pair of geometrical elements. The problem is to find the optimal domain approximately for a given functional that is involved with the solution of a linear or nonlinear elliptic equation with a boundary condition over a domain. The Shape-Measure method, in Cartesian coordinates, will be used to find the nearly optimal solution in two steps. By transferring the problem into a measure-theoretical form, first we will find the solution of the elliptic problem for a given domain by using the embedding method. Then the Shape-Measure method will be applied to find the best domain approximately. An example will be given.

1 Introduction and Problem

Consider the optimal shape (optimal shape design) problems in which they are defined with respect to a pair of geometrical elements; this pair consists of a measurable set (in $I\!R^2$), which can be regarded as a domain, and a simple closed curve containing a given point, which is the boundary of the set. By considering the property for the desired curves to be simple, the problem depends on the geometry which is used. In polar coordinates, we solved the similar problem in [1]. But in Cartesian coordinates, it is difficult to introduce a linear condition which determines the property of a closed curve being simple. Thus here we consider some limitation on shape in order to make sure that it is simple. The problem will be solved in two stages. First, by use of measures, the value of the objective function will be calculated for any given domain. Then the optimal domain will be obtained by use of optimization techniques.

Let $D \subset I\!R^2$ be a bounded domain with a piecewise-smooth, closed and simple boundary ∂D. We assume that some part of ∂D is fixed and the rest, Γ, with the given initial and final points A and B respectively, is not fixed. Here we suppose that the fixed part of ∂D is made by three segments, parts of lines $y = 0, x = 0$ and $y = 1$ between points $A(1,0), (0,0), (0,1), B(1,1)$ (see Figure 1).

Thus Γ is chosen as an appropriate variable curve joining A and B so that D is well-defined. Let $u(X)$ $(X = (x,y) \in I\!R^2)$ be a bounded solution of the following elliptic

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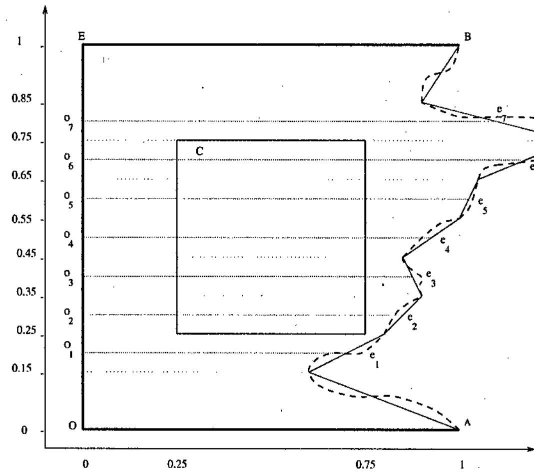

FIG. 1. An admissible domain D under the assumptions of the numerical work.

equation:

$$\Delta u(X) + f(X,u) = v(X)\ ,\ u_{|\partial D} = 0, \tag{1.1}$$

where $X \in D \to v(X) \in \mathbb{R}$ is a bounded real function (v also can be considered as a fixed control function); the function f is assumed to be a bounded and continuous real-valued function in $L_2(D \times \mathbb{R})$. Moreover the above domain D is called an *admissible* if the equation (1.1) has a bounded solution on D; we denote by $\mathcal{D}$ as the set of all such admissible domains. We are going to solve the problem of minimizing the functional $\mathbf{I}(D) = \int_D f_\circ(X,u)\,dX$, on the set $\mathcal{D}$ where $f_\circ$ is a given continuous, nonnegative, real-valued function on $D \times \mathbb{R}$. To calculate the value of $\mathbf{I}(D)$ for a given domain D, it is necessary first to identify the solution of (1.1).

2 Weak solution and metamorphosis

In general, it is difficult to identify a classical solution for the problem like (1.1) and usually one tries to find a *weak* (generalized) solution of them. Hence the variational form of (1.1) is introduced in the following; we remind the reader that $H_0^1(D) = \{\psi \in H^1(D) \ :\ \psi_{|\partial D} = 0\}$, where $H^1(D)$ is the Sobolev space of order 1.

Proposition 2.1 *Let u be the classical solution of (1.1), then we have the following equality:*

$$\int_D (u\Delta\psi + \psi f)\,dX = \int_D \psi v\,dX\ ,\ \forall\psi \in H_0^1(D). \tag{2.1}$$

Proof: Multiplying (1.1) by the function $\psi \in H_0^1(D)$ and then integrating over D, with use of the Green's formula (see [3]) gives $\int_D (u\Delta\psi + \psi f - \psi v)\,dX = \int_{\partial D}(\psi\frac{\partial u}{\partial \mathbf{n}} - u\frac{\partial \psi}{\partial \mathbf{n}})\,dS$, where $\mathbf{n}$ is the unit vector normal to the boundary ∂D and directed outward with respect to D. Because $\psi_{|\partial D} = 0$ and $u_{|\partial D} = 0$, then (2.1) is satisfied. □

Definition 2.2 *A function $u \in H^1(D)$ is called a bounded weak solution of the problem (1.1) when it is bounded and satisfies the equality (2.1) for all $\psi \in H_0^1(D)$ (the conditions for existence of the weak solution of the problem (1.1) and also the boundedness property of it, have been considered in many references, like [3] and [2]).*

Now we apply our new way which is called the *Shape-Measure* method. Let $\Omega \equiv U \times \overline{D}$, where $U \subset I\!R$ is the smallest bounded set in which the bounded weak solution $u(\cdot)$ takes values. Then by applying the Riesz Representation Theorem ([6]), a bounded weak solution can be represented by a positive Radon measure; the proof of the following Proposition is similar to the Proposition 3.1 in [1].

Proposition 2.3 *Let $u(X)$ be a bounded generalized solution of (1.1); there exist a unique positive Radon measure, say μ_u, in $\mathcal{M}^+(\Omega)$ such that:*

$$\mu_u(F) \equiv \int_\Omega F\, d\mu_u = \int_D F(X,u)\, dX\ ;\ \forall F \in C(\Omega). \tag{2.2}$$

Thus the equality (2.1) can be changed to $\mu_u(F_\psi) = \gamma_\psi$, $\forall \psi \in H_0^1(D)$, where $F_\psi = u\Delta\psi + f\psi$ and $\gamma_\psi = \int_D \psi v\, dX$. Also, $\mathbf{I}(D) = \mu_u(f_\circ)$. Because the measure μ_u projects on the (x,y)-space as the respective Lebesgue measure, we should have $\mu_u(\xi) = a_\xi$, where $\xi : \Omega \to I\!R$ depends only on variable X (i.e. $\xi \in C_1(\Omega)$), and a_ξ is the Lebesgue integral of ξ over D. Therefore the original problem can be described as follows:

To find a measure $\mu_u \in \mathcal{M}^+(\Omega)$ so that it satisfies the following constraints:

$$\begin{aligned} \mu_u(F_\psi) &= \gamma_\psi, && \forall \psi \in H_0^1(D); \\ \mu_u(\xi) &= a_\xi, && \forall \xi \in C_1(\Omega). \end{aligned} \tag{2.3}$$

As Rubio did in [5], to be sure that we do not miss any solution, we extend the underlying space; instead of finding a measure $\mu_u \in \mathcal{M}^+(\Omega)$, introduced by Proposition 2.3 and equalities (2.3), we seek a measure $\mu \in \mathcal{M}^+(\Omega)$ which satisfies just the conditions:

$$\begin{aligned} \mu(F_\psi) &= \gamma_\psi, && \forall \psi \in H_0^1(D); \\ \mu(\xi) &= a_\xi, && \forall \xi \in C_1(\Omega). \end{aligned} \tag{2.4}$$

3 Approximation

The system (2.4) is linear because all the functions in the right-hand-side of equations are linear functions in their argument μ. But the number of equations and the underlying space are not finite. We shall develop this system by requiring that only a finite number of the constraints are satisfied. This will be achieved by choosing countable sets of functions whose linear combinations are dense in the appropriate spaces. But first we should approximate the unknown part of the boundary just by the finite number of its points. This idea comes from the approximation of a curve by broken lines. For the given D and hence for the given Γ, let $A_m = (x_m, y_m), m = 0, 1, 2, \ldots, M$, be a finite number of points on Γ (where $A_0 = A$). We link together each pair of consecutive points A_m and A_{m+1} for $m = 0, 1, \ldots, M-1$ and close this curve by joining the points A_M and B together. Now the resulted shape, which is denoted by ∂D_M, is an approximation for

∂D; we also call the domain which introduced by its boundary ∂D_M as D_M (see Figure 1).

It is possible that by increasing M, the curve ∂D_M will become closer and closer (in the Euclidean metric) to the curve ∂D, and hence one may conclude that the minimizer of $\mathbf{I}$ over $\mathcal{D}_M$, if it exists, tends to the minimizer of $\mathbf{I}$ over $\mathcal{D}$, if it exists. But some difficulties could arise (too oscillatory a curve may cause problems). Thus, we will fix the number of points. For a given M, let the value of the components $y_1, y_2, \ldots, y_M$, be fixed. Because x_m is a free term, the point A_m could be anywhere on the line $y = Y_m, x \geq 0$ for every m (see Figure 1). Therefore points A_m and A_{m+1} can be chosen so that they belong to Γ and hence the part of Γ between the lines $y = Y_m$ and $y = Y_{m+1}$ can be approximated by the segment $A_m A_{m+1}$. Hence, we do not lose generality. Thus, we fix the components $y_1, y_2, \ldots, y_M$ with the values $Y_1, Y_2, \ldots, Y_M$, respectively.

Now we introduce the set $\{\psi_i \in H_0^1(D) : i = 1, 2, \ldots\}$ so that the linear combinations of the functions $\{\psi_i\}$ are uniformly dense (that is, dense in the topology of the uniform convergence) in $H_0^1(D)$. We know that the vector space of polynomials with the variable x and y, $P(x, y)$, is dense in $C^\infty(\overline{D})$; therefore the set $P_0(x, y) = \{p(x, y) \in P(x, y) \mid p(x, y) = 0, \forall (x, y) \in \partial D\}$, is dense (uniformly) in $\{h \in C^\infty(\overline{D}) : h_{|\partial D} = 0\} \equiv C_0^\infty(\overline{D})\}$. Since the set

$$Q(x, y) = \{1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3, \ldots\}$$

is a countable base for the vector space $P(x, y)$, each elements of $P(x, y)$ and also $P_0(x, y)$, is a linear combination of the elements in $Q(x, y)$. By Theorem 3 of Mikhailov [3] page 131, the space $C^\infty(\overline{D})$ is dense in $H^1(D)$; thus the space $C_0^\infty(\overline{D})$ will be dense in $H_0^1(D)$. Consequently, the space $P_0(x, y)$ is uniformly dense in $H_0^1(D)$. We define

$$\psi_i(x, y) = xy(y-1)\prod_{l=1}^{M}(x - x_l + y - Y_l)q_i(x, y), \tag{3.1}$$

where $q_i \in Q(x, y)$. Therefore $\psi_{|\Gamma} = 0$ and the set $\{\psi_i(x, y) : i = 1, 2, \ldots\}$, is total (dense in the topology of the uniform convergence) in $H_0^1(D)$.

For the second set of functions, let L be a given positive integer and divide D into L (not necessary equal) parts $D_1, D_2, \ldots, D_L$, so that by increasing L the area of $D_s, s = 1, 2, \ldots, L$, will be decreased. Then, for each s we define:

$$\xi_s(x, y, u) = \begin{cases} 1 & \text{if } (x, y) \in D_s, \\ 0 & \text{otherwise.} \end{cases}$$

These functions are not continuous, but each of them is the limit of an increasing sequence of positive continuous functions, $\{\xi_{s_k}\}$; then if μ is any positive Radon measure on Ω, $\mu(\xi_s) = \lim_{k \to \infty} \mu(\xi_{s_k})$. The linear combination of functions $\{\xi_j : j = 1, 2, \ldots, L\}$ for all positive integer L, can approximate a function in $C_1(\Omega)$ arbitrary well (see [5] Chapter 5).

By selecting just the finite number of functions in the mentioned spaces the problem (2.4) can be replaced by another one in which we are looking for the measure $\mu_{M_1, M_2} \in$

$\mathcal{M}^+(\Omega)$, so that it satisfies the following constraints:

$$\begin{aligned} \mu_{M_1,M_2}(F_i) &= \gamma_i, \qquad i = 1,2,\dots,M_1;\\ \mu_{M_1,M_2}(\xi_j) &= a_j, \qquad j = 1,2,\dots,M_2, \end{aligned} \tag{3.2}$$

where M_1 and M_2 are two positive integers and $F_i \equiv F_{\psi_i}$, $\gamma_i \equiv \gamma_{\psi_i}$, $a_j \equiv a_{\xi_j}$. If we denote by $Q(M_1, M_2)$ the set of positive Radon measures in $\mathcal{M}^+(\Omega)$ which satisfy equalities (3.2), and also denote by Q the set of positive Radon measures in $\mathcal{M}^+(\Omega)$ which satisfy equalities (2.4), one can easily prove the following Proposition by considering the proof of Proposition $III.1$ in [5].

Proposition 3.1 : *If $M_1, M_2 \longrightarrow \infty$ then $Q(M_1, M_2) \longrightarrow Q$; hence for the large enough numbers M_1 and M_2 the set Q can be identified by $Q(M_1, M_2)$.*

But even if the number of equations in (3.2) is finite, the underlying space $Q(M_1, M_2)$ is still infinite-dimensional. By Theorem $A.5$ in the Appendix of [5], μ_{M_1,M_2} in (3.2) can be characterized as $\mu_{M_1,M_2} = \sum_{n=1}^{M_1+M_2} \alpha_n \delta(Z_n)$, with triples $Z_n \in \Omega$ and the coefficients $\alpha_n \geq 0$ for $n = 1, 2, \dots, M_1 + M_2$, where $\delta(z) \in \mathcal{M}^+(\Omega)$ is supposed to be a unitary atomic measure with support the singleton set $\{z\}$. Thus the measure problem is equivalent to a nonlinear one in which the unknowns are the coefficients α_n and supports $\{Z_n\}$. Proposition $III.3$ of [5] Chapter 3, states that the measure μ_{M_1,M_2} has the following form

$$\mu_{M_1,M_2} = \sum_{n=1}^{N} \alpha_n \delta(Z_n), \tag{3.3}$$

where $Z_n, n = 1, 2, \dots, N$, belongs to a dense subset of Ω. Now let us put a discretization on Ω, with the nodes $Z_n = (x_n, y_n, u_n)$, in a dense subset of Ω; then we can set up the following linear system in which the unknowns are the coefficients α_n:

$$\begin{aligned} \alpha_n &\geq 0, \qquad n = 1,2,\dots,N;\\ \sum_{n=1}^{N} \alpha_n F_i(Z_n) &= \gamma_i, \qquad i = 1,2,\dots,M_1;\\ \sum_{n=1}^{N} \alpha_n \xi_j(Z_n) &= a_j, \qquad j = 1,2,\dots,M_2. \end{aligned} \tag{3.4}$$

The solution of (3.4) is not necessary unique (even if the problem (1.1) satisfies the necessary conditions for having a unique bounded weak solution), because of the approximation scheme.

4 The optimal solution

The main aim of the present section is to find an optimal domain $D^* \in \mathcal{D}_M$ so that the value of $\mathbf{I}(D^*)$ will be the minimum on the set $\mathcal{D}_M$. By applying the result of the previous section, a solution of (1.1) can be found. Indeed, it is approximated by a solution of the linear system (3.4) according to the variables, $x_m, m = 1, 2, \dots, M$. As mentioned, this solution is not necessary unique. Let us specify one by solving the following linear programming problem

$$
\begin{array}{lll}
\textbf{Minimize :} & \displaystyle\sum_{n=1}^{N} \alpha_n f_\circ(Z_n) & \\
\textbf{Subject to :} & \alpha_n \geq 0, & n = 1, 2, \ldots, N; \\
& \displaystyle\sum_{n=1}^{N} \alpha_n F_i(Z_n) = \gamma_i, & i = 1, 2, \ldots, M_1; \\
& \displaystyle\sum_{n=1}^{N} \alpha_n \xi_j(Z_n) = a_j, & j = 1, 2, \ldots, M_2.
\end{array}
\tag{4.1}
$$

Thus, for each D, the value $\mathbf{I}(D) = \int_D f_\circ(X, u)\, dX \equiv \mu(f_\circ) \simeq \mu_{M_1,M_2}(f_\circ)$, is defined uniquely in terms of the variables $x_m, m = 1, 2, \ldots, M$. So, we set up a function, $\mathbf{J}$, on $\mathcal{D}_M$ defined by $D \in \mathcal{D}_M \longrightarrow \mathbf{I}(D) \cong \mu_{M_1,M_2}(f_\circ)$ where $\mu_{M_1,M_2}(f_\circ) = \sum_{n=1}^{N} \alpha_n f_\circ(Z_n)$. Clearly $\mathbf{J}$ can be regarded as a vector function:

$$\mathbf{J} \; : \; (x_1, x_2, \ldots, x_M) \in I\!R^M \longrightarrow \mu_{M_1,M_2}(f_\circ) \in I\!R. \tag{4.2}$$

Since $\mathbf{J}$ is a real-valued function which is bounded below, and is defined on a compact set (since constraints are to be put in the variables), it is possible to find a sequence of points so that the value of the function along the sequence tends to the (finite) infimum of the function. The coordinate values corresponding to the points in the sequence are of course finite. Now, suppose that $(x_1^*, x_2^*, \ldots, x_M^*)$ is the minimizer of the vector function $\mathbf{J}$; it can be identified by using one of the related minimization methods. The introduced domain by the minimizer is denoted by D^*. We assume in the following theoretical result that the minimization algorithm which is used, is perfect; that is, it comes out with the *global minimum* of J in its (compact) domain.

Theorem 4.1 **:** *Let M, M_1 and M_2 be the given positive integers which were defined in section 3, and D^* be the minimizer of (4.2) as mentioned above. Then D^* is the minimizer domain of the functional $\mathbf{I}$ over $\mathcal{D}_M$ and the value of $\mathbf{I}(D^*)$ can be approximated by $\mathbf{J}(D^*)$; moreover $\mathbf{J}(D^*) \longrightarrow \mathbf{I}(D^*)$ as M_1 and M_2 tend to infinity.*

Proof: Suppose D^* is not the minimizer of $\mathbf{I}$; hence there exists a domain, call D', in $\mathcal{D}_M$ so that $\mathbf{I}(D') < \mathbf{I}(D^*)$. Proposition 2.3 shows that there is a unique measure, call μ', in $\mathcal{M}^+(\Omega)$ so that $\mathbf{I}(D') = \mu'(f_\circ)$, and also Proposition 3.1 states that for sufficiently large numbers M_1 and M_2, $\mu'(f_\circ)$ can be approximated by $\mu'_{M_1,M_2}(f_\circ)$ in $Q(M_1, M_2)$. Thus, $\mathbf{I}(D') \cong \mu'_{M_1,M_2}(f_\circ) = \mathbf{J}(D')$. In the same way, one can show that $\mathbf{J}(D^*)$ approximates $\mathbf{I}(D^*)$; so $\mathbf{I}(D^*) \cong \mu^*_{M_1,M_2}(f_\circ) = \mathbf{J}(D^*)$. Hence $\mathbf{J}(D') < \mathbf{J}(D^*)$. which is contrary with the fact that D^* is the minimizer of $\mathbf{J}$. Moreover, from Proposition 3.1 it follows that $\mathbf{J}(D^*)$ tends to $\mathbf{I}(D^*)$ as $M_1, M_2 \longrightarrow \infty$. □

5 Numerical example

We consider the elliptic equations (1.1) with

$$v(x,y)=\begin{cases}1 & \text{if } (x,y)\in D\cap C,\\ 0 & \text{otherwise,}\end{cases}$$

where C is the square $[\frac{1}{4},\frac{3}{4}]\times[\frac{1}{4},\frac{3}{4}]$ (see Figure 1). We also take $M=8$, $M_1=10$, $M_2=8$, $N=740$ (the 36 number of nodes are chosen so that $u_{|\partial D}=0$) and suppose $Y_1,Y_2,\dots,Y_8$ are $0.15,0.25,\dots,0.85$, respectively. By extra constraints, $x_m\geq\frac{3}{4}$, $m=2,3,\dots,7$, the value of γ_i for any $D\in\mathcal{D}_M$ is defined as $\gamma_i=\int_{\frac{1}{4}}^{\frac{3}{4}}\int_{\frac{1}{4}}^{\frac{3}{4}}\psi_i(x,y)\,dxdy$, $i=1,2,\dots,10$. We also assume that the function u takes values in U =[-1,1], and consider the polynomials $q_i(x,y)$ as $1,x,y,x^2,xy,y^2,x^3,x^2y,xy^2,y^3$. The function $f_\circ$ is chosen as $f_\circ=(u-0.1)^2$. This function can be considered as a distribution of heat in the surface for the system governed by an elliptic equations.

In minimization, we apply the Downhill Simplex Method in Multidimension by using the Subroutine *AMOEBA* (see [4]) and also consider an upper bound for variables (suppose they are not higher than 2). These conditions are applied by means of the penalty method (see [7]). Hence, for the nonlinear case of the partial differential equations (1.1), we have taken $f(x,y,u)=0.25u^2$, and used the initial values as $X_m=1.0, m=1,2,\dots,8$, and the stopping tolerance for the program (variable *ftol* in the Subroutine *AMOEBA*) as 10^{-7}. We remind the reader that the functions F_i and the values of γ_i, $i=1,2,\dots,10$, have been calculated by the package "*Maple V.3*". The results are: the optimal value of $\mathbf{I}$ = 0.45467920356379, the number of iterations = 502, the value of the variables in the final step are $X_1=1.05019$, $X_2=1.08521$, $X_3=0.750001$, $X_4=0.768701$, $X_5=1.12986$, $X_6=1.13775$,$X_7=0.97783$, $X_8=1.61566$, which represent the optimal domain, shown in the Figure 2.

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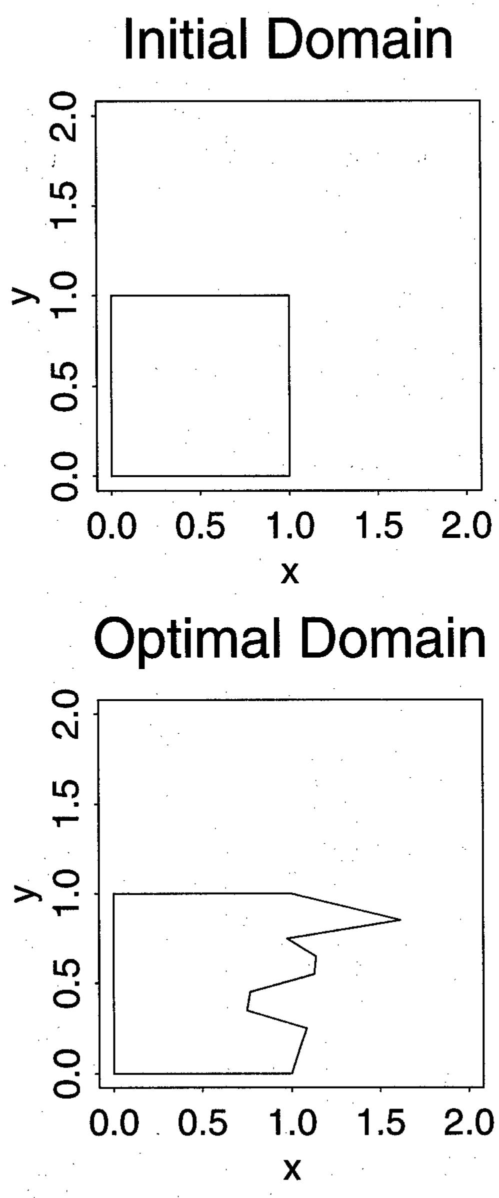

FIG. 2. The initial and the optimal domain for nonlinear case of elliptic equations.